WiSe 24/25

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The Four Colour Theorem

Any connected, planar, simple graph can be vertex-coloured with only four colours so that no two vertices can share an edge and a colour at once.

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Submission date: 11.11.24

Introduction

The four colour theorem is one of those problems with a very interesting history. After its discovery in 1852 by Francis Guthrie it was allegedly proven by Alfred Kempe in 1879 [1]. The mathematical community accepted the proof for almost 11 years, before, mournfully, Percy Heawood found a flaw in Kempe's proof [2]. It took almost a century for a new generation of mathematicians to finally prove the theorem in 1976, and that, only under heavy computational assistance [2]: It is, until this day, a theorem that has been verified only by computers, although within different methods [3].

Of course, many papers have been published, claiming that they have finally found a human-verifiable proof, but none of them has been accepted by the mathematical community, yet [4–7].

Since the theorem is reasonably famous for this history, and its insights on map-colouring are fun to explore, (not only for the mathematician,) the theorem has been explained in many resources already [8–10].

In fact, manually writing this introduction was superfluous in times of Co-Pilot writing it *on the fly* with the same depth of information (see Appendix A). Therefore, I will focus in this report on my personal approach of explaining the key ideas necessary to prove the theorem which remains a unique contribution to the World Wide Web.

Contents

The proofs main ideas

With the common notation for the sets of vertices V, edges E and faces F, let $G(V, E, F)$ be a connected, planar, simple graph, that is,

- There is always a path of edges from any vertex to any other in G,
- It is always possible to embed G in the plane without any edges crossing,
- There are no loops, no multiple or directed edges in G.

Let $\mathcal{G}$ be the set of all such graphs G and note that for any planar graph Euler's formula $|V| - |E| + |F| = 2$ holds [2].

In the following report we will use the notation $v \in G :\Leftrightarrow v \in V$.

Def. k-colouring

For a set of colours $C \in \mathbb{N}_+^{\leq k}$ and a colouring $\mathcal{C} : V \longrightarrow C$, G is said to have a k-**colouring** iff $|\operatorname{im}(\mathcal{C})| = k$ and for any two vertices $v_1, v_2 \in V$ with $(v_1, v_2) \in E$ it holds that $\mathcal{C}(v_1) \neq \mathcal{C}(v_2)$.

Note that it could be possible that a graph G has multiple such colourings $\mathcal{C}$, even if we remove all those which result due to permutation of colour labels in C. Therefore, let $\mathfrak{C}_k := \{\mathcal{C} | \mathcal{C} \text{ is a } k\text{-colouring of } G\}$ be the set of all such colourings of G.

Induction

We proceed with an induction over the amount of vertices of G.

Initial step: Let $|V| \leq 4$ then any G with this number is obviously 4-colourable.

Induction step: Assume that all graphs G with $|V| \leq n$ are 4-colourable. Mission is now to prove that any G with $|V| = n + 1$ vertices is also 4-colourable.

Consider the graph G with $|V| = n + 1$ in the induction step. If we split the graph into two **proper** (connected, planar) subgraphs G_1, G_2 so that $G_1 \cup G_2 = G$, then $|V_1|, |V_2| < n + 1 \Rightarrow |V_1|, |V_2| \leq n$ and, thus, by means of our induction step assumption, both G_1, G_2 are 4-colourable. If we could find a 4-colouring $\mathcal{C}_1, \mathcal{C}_2$ for each subgraph so that those vertices that overlap share the same colour, G is 4-colourable, since there is no conflict between $\mathcal{C}_1(G_1)$ and $\mathcal{C}_2(G_2)$ in the intersection $G_1 \cap G_2$:

$$\mathcal{C}_1(G_1 \cap G_2) = \mathcal{C}_2(G_1 \cap G_2)$$

$$\Rightarrow \mathcal{C}(v) := \begin{cases} \mathcal{C}_1(v) \text{ if } v \in G_1 \\ \mathcal{C}_2(v) \text{ else} \end{cases}$$

Def. Triangulation

Studying these graphs and some examples, Kempe observed:

- A **triangulation**, that is a planar graph in G to which no edge can be added without breaking planarity, occupies the largest possible number of restrictions for a vertex-colouring.
- It is, therefore, only necessary to prove the four colour theorem on triangulated graphs. Let the set of triangulations be $\mathcal{G}_\Delta \subset \mathcal{G}$.
- Precisely, the number of edges in a triangulation depends on the number of vertices and vise versa: $|E| = 3|V| - 6$ for all $|V| \geq 3$ [2].

Def. Unavoidable Set

Additionally, Kempe observed, that it is difficult to construct a graph in $\mathcal{G}$ that has vertices of $\deg(v) > 5$ only, where $\deg : V \to \mathbb{N}_+$ counts the number of connected edges to $v \in V$. In fact, it is impossible:

Assume $\forall v \in V : \deg(v) > 5$ then $\deg(v) \geq 6$. Consequently,

$$6 \cdot |V| \leq \sum_{v \in V} \deg(v) = 2\ |E|$$

$$2\ |E| = 6|V| - 12 \quad \text{(see section above)}$$

Putting the two together yields $0 \leq -12$ proving by contradiction that there must be at least one vertex v with $\deg(v) \leq 5$.

Let this vertex v be labeled $v_{\deg = k}$ for some $k \leq 5$, then the subgraph consisting of the neighbouring vertices and $v_{\deg = k}$ is labeled $R_I(\{v_{\deg = k}\}) \subseteq G$. The notation is properly explained in the next section.

The results show, that it is impossible for all $G \in \mathcal{G}_\Delta$ that it does not contain any of the elements of

$$U' = \{R_I(\{v_{\deg = k}\}) | k \in \mathbb{N}_+^{\leq 5}\}$$

More generally, an **unavoidable set** U is a set of graphs in $\mathcal{G}_\Delta$ for which it is true that it is impossible for any $G \in \mathcal{G}_\Delta$ to not contain any of the elements of U:

$$\forall G \in \mathcal{G}_\Delta : \exists g \in U : g \subseteq G$$

U' is an example for such a set U.

Def. Outer and Inner Graph

Recall the idea from the section "Induction" that if we can split any $G \in \mathcal{G}_\Delta$ in two proper subgraphs and prove that we can 4-colorize both so that the overlapping vertices share the same colours, we have found a 4-colouring of G.

Note that it might not be necessary to know the exact structure of both subgraphs to prove that their intersection matches.

To keep the intersection as small as possible, we introduce the idea of the outer and inner graph:

Let $G \in \mathcal{G}_\Delta, g \subset G$ and $g \in \mathcal{G}_\Delta$, define the **outer graph of** g as $R_O(g) \coloneqq G \setminus g$ and the **inner graph of** g as $R_I(g) \coloneqq \{v \in G | \exists w \in g : (v, w) \in E\}$.

In most cases, the overlap $R(g) \coloneqq R_I(g) \cap R_O(g) = R_I(g) \setminus g$ is, due to triangulation, exactly a k-cycle of vertices for some $k \in \mathbb{N}_+$. Special cases are comprehensively elaborated in the appendices of [2] and can be excluded.

Since G could contain any number of vertices we cannot iterate over all possible colourings. However, since we know G contains one of the elements g of U, we can iterate over all possible colourings of $R(g)$ and see if we can find a strategy to extend that colouring into the finite graph g. This leads us to the term:

Def. Reducible

A subgraph g in G is called **reducible** if there is a strategy to extend any colouring of $R(g)$ into g.

This is very wage since neither strategy nor extending is defined properly, yet. However, we need some more tools from Kempe to clarify this term.

The important note is, if $\forall g \in U : g$ is reducible (and U is a valid unavoidable set for all $G \in \mathcal{G}_\Delta$), then we have effectively proven the Four Colour Theorem.

Def. Kempe Chain / Kempe Component

A subgraph $\text{Kemp}_{a,b}(v) \subseteq G$ for $a, b \in C, v \in V$ is called **Kempe chain** or **Kempe component** under the fixed colouring $\mathcal{C} \in \mathfrak{C}_k$ iff it is the maximal connected subgraph of G consisting only of vertices coloured with a or b, and including v:

$$\text{Kemp}_{a,b}(v) = \max\{G' \subseteq G | G' \text{ connected}, v \in G', \forall w \in G' : \mathcal{C}(w) \in \{a, b\}\}$$

The max-function returns the graph with the largest number of vertices and edges, removing those edges with loose ends.

Def. Partition and Kempe-Partition

A set of disjunct two-element colour sets, which, in union, equal C, is called a **partition of** C. Consequently, it is only possible to obtain a partition when $|C|$ is even. The set of **all possible partitions** is

$$P(C) := \left\{ \left\{ s_1, s_2, ..., s_{\frac{|C|}{2}} \right\} | \bigcup_{i=1}^{\frac{|C|}{2}} s_i = C, |s_i| = 2 \right\}$$

As example, $\{\{1, 2\}, \{3, 4\}\}$ or $\{\{1, 3\}, \{2, 4\}\}$ are both partitions of $C = \{1, 2, 3, 4\}$.

The partition of C can be used to create a partition of G. A set K of subgraphs of G that are all Kempe components, is called **Kempe-Partition relative to** a partition $p \in P(C)$:

$$K_p := \left\{ \text{Kemp}_{a,b}(v) | \{a, b\} \in p, v \in G \right\}$$

Def. Kempe Exchange

Kempe was the first known to observe that in a 4-colouring the two colours of any Kempe component $\mathtt{k} \in K_p$ could be swapped without changing K_p presumed that p is not changed.

His erroneous proof failed because he changed p assuming that it would not affect K_p and continued his colour exchanges on previous Kempe components which could be different after the swap.

Formally, a **Kempe exchange** $\text{Ex}[(a, b), v] : \mathfrak{C}_{|C|} \to \mathfrak{C}_{|C|}$ defined with

$$\text{Ex}[(a, b), v](\mathcal{C})(w) := \begin{cases} \mathcal{C}(w) \text{ if } w \notin \text{Kemp}_{a,b}(v) \text{ , else} \\ a \text{ if } \mathcal{C}(w) = b \\ b \text{ if } \mathcal{C}(w) = a \end{cases}$$
$$\text{for } (a, b) \in p \in P(C), v \in G$$

is a map that swaps the colours of $\text{Kemp}_{a,b}(v)$.

The first observation is:

If $\mathcal{C}$ is a valid 4-colouring of G, then for a fix partition $p \in P(C)$, so is $\text{Ex}[(a, b), v](\mathcal{C})\ \forall \{a, b\} \in p, v \in G$.

This is important because if we cannot prove reducibility for some colouring $\mathcal{C}$, we can try again on an alternative colouring deduced from the original using Kempe exchanges. See more in [section "Alternative Colouring"].

Def D-reducible

A subgraph g in G is called **D-reducible** if there is a strategy to extend **any** colouring of $R(g)$ into g, that is, for all colourings $\mathcal{C} \in \mathfrak{C}_4$ of $R(g)$ there exists an alternative colouring $\mathcal{C}_A$ of $R(g)$ and a 4-colouring $\mathcal{C}_E$ of $R_I(g)$ so that $\forall v \in R(g) : \mathcal{C}_A(v) = \mathcal{C}_E(v)$.

So there is a finite number of $\mathcal{C}_E$ determined by g.

And there is a finite number of $\mathcal{C}_A$ determined by k, the number of vertices in $R(g)$.

If we find this pair of colourings for each colouring $\mathcal{C}$ and for each element of the finite set U, we have proven the theorem.

Def. Alternative Colouring

Let $\mathcal{C}$ be the colouring of the k-cycle $R(g)$. An **alternative colouring** $\mathcal{C}_A$ is a colouring obtained by Kempe exchanges on the colouring of $R_O(g)$. Although we do not know the structure of $R_O(g)$, we have the attribute of Kempe exchanges never changing the validity of $|C|$-colourings. This will aid us permute the colours on $R(g)$ but being rest assured that $R_O(g)$ remains $|C|$-colourable.

Note, that if g is not *D-reducible*, it does not mean there is no other strategy to extend the outer colouring into g, possibly making g reducible in some other way. In fact, Appel and Haken [2], also made use of C-reducibility which uses other arguments to prove extendability of any outer colouring.

An alternative colouring can be obtained by the following procedure:

Let the vertices in $R(g)$ be labeled clockwise in order $x_1, ..., x_k$. In the following section always regard the index of x under subscript arithmetic modulo, since the choice of x_1 is arbitrary.

1. Fix any disjunct subsets $F_1, ..., F_m \subset R(g)$ that have the following attributes:
   1. $$\bigcup_{i \in [m]} F_i = R(g)$$
   2. $$\exists p \in P(C) : \forall i \in [m] :$$
      1. $$\exists \{a, b\} \in p : \forall x \in F_i : \mathcal{C}(x) \in \{a, b\}$$
      2. $$\forall x_l \in F_i : (\mathcal{C}(x_{l\pm1}) \in \{a, b\} \Rightarrow x_{l\pm1} \in F_i)$$
      3. $\forall x_p, x_q \in F_i$ for $p < q : \nexists r, t$ with $p < r < q < t$ and $x_r, r_t \in F_j$ for some $j \neq i$
2. Note that $F_1, ..., F_m$ are none overlapping Kempe chains so we can define a Kempe exchange over the F_i's ($i \in [m]$):
   $$\text{Ex}[F_i] := \text{Ex}[(a, b), v] \text{ for some } v \in F_i, a, b \in \mathcal{C}(F_i), a \neq b$$
3. Get an alternative colouring by applying any Kempe exchanges that are allowed within the fixed Kempe connections $F_1, ..., F_m$:
   $$\mathcal{C}_A \in \left\{\text{Ex}\left[F_{i_1}\right] \circ ... \circ \text{Ex}\left[F_{i_z}\right](\mathcal{C}) \mid i_1, ..., i_z \in [m]\right\}$$

Discharging Argument

Unfortunately, while U' is unavoidable, nobody was yet able to prove the reducibility of $R_I(\{v_{\deg=5}\})$ (Kempe succeeded, however, in showing reducibility for all $R_I(\{v_{\deg=k}\})$ with $k < 5$ [1]).

So, the mathematical community was in need of finding more complex U that is at the same time fully reducible.

What has been used is a technique that offers enormous creativity in the way it is applied because it offers numerous possible adjustment [2]. In fact, this has led improvements on the size of U in past papers [2, 11].

The principle shall be outlined only briefly since it has been widely explained in other resources [9, 8] already.

Let $G \in \mathcal{G}_\Delta$ have an attribute f. Typically, this is called the **charge of the graph**. Introduce a set of logical rules L that can be applied on G and prove that they will always keep the attribute f constant. Typically, the charge f is a positive natural number.

Then apply some of these rules on those vertices that are charged positively and analyse all those vertices after this, so called, discharging. There must be some vertices having a positive charge, otherwise it would be a contradiction to the attribute of rules in L that they always retain charge and f being positive.

Consequently, these vertices cannot be avoided and, depending on the rule set, all those vertices connected to them can neither. They form a subgraph and all possible such subgraphs that retain the charge form an unavoidable set U_L depending on the rule set L.

Simple examples are shared in many sources (for instance [8]), the original proof, however, does not use less than 300 rules in their highly specialised discharging method [2, 8]

Appendix A

The following paragraph has been fully generated by Co-Pilot while trying to write an introduction to the report. It shows that there is no longer need for repeating what has been said many times:

> The four colour problem is a classic problem in mathematics. It asks whether it is possible to colour any map in such a way that no two adjacent regions have the same colour, using only four colours. The problem was first posed in 1852 by Francis Guthrie, a student of Augustus De Morgan. After many failed attempts to prove the conjecture, it was finally proven in 1976 by Kenneth Appel and Wolfgang Haken under heavy use of computer assistance. This made it the first problem ever to be solved with the aid of computation. In this report, we will discuss the key ideas that led to the successful proof of the four colour theorem.

Appendix B

I, personally, would be sad not to point out that Appel and Haken have been writing their complete proof of over 1100 pages (400 pages alone are computer output of graphs that were checked for reducibility) with exuberant humor, e.g.

- On the remark that with their technique a large number of proofs of the Four Colour Theorem can be found they say: "[Even] one proof of this type is probably one more than many people really want to see." [2]
- In fact, it is possible to manually verify reducibility for all graphs in U, "[…] with only a few years of careful work. There are obviously some slackers who would not be fascinated by such a task. Such people, with an immorally low tolerance for honest hard work, tend to program computers to do this task." [2]
- And, finally, they also admitted that, "One can never rule out the chance that a short proof of the Four-Color Theorem might some day be found, perhaps by the proverbial bright high-school student." [2]

Appendix C

Note to the course instructor: This report reveals the main aspects that I have learned but certainly not *all* the things I have learned. Additionally, I formalized many terms in my style, so it does not repeat the many resources that can be found on this topic. What I excluded from this short report is:

- Proper proofs of theorems and observations
- Spherical projection
- Historical elaboration
- Concrete examples as given in the presentation

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